CRYO/06/034
September, 30, 2006

MAGNUM - A Fortran Library for the Calculation of Magnetic Configurations

L. Bottura

Distribution: P. Bruzzone, M. Calvi, J. Lister, C. Marinucci (EPFL/CRPP), A. Portone (EFDA-CSU)

Keywords: Magnetic field

Summary

This note reports general formulae for the calculation of magnetic field generated by conductors in plane 2-D, axisymmetric 2-D and 3-D configurations, in the presence of current and magnetization sources. All formulae have been programmed in numerically stable routines, collected in a library named MAGNUM.

Introduction

In this note we review the state of the art of integral formulae for the calculation of the magnetic field in linear and magnetised media. The formulae have been collected in a library of FORTRAN routines that compute the magnetic field and the vector potential generated by distributions of current and magnetization in several configurations of interest for magnet design and analysis. The routines are organised according to the field source (current or magnetization), and spatial symmetry, for the following configurations:

- plane 2-D
- axisymmetric 2-D
- general 3-D

Most of the calculations are based on analytical formulae for the field, and numerical integration or function approximation is limited to the minimum strictly necessary. In particular, all plane 2-D and 3-D configurations are computed analytically. Axisymmetric 2-D configurations require a numerical approximation to elliptic integrals and numerical integration in one dimension.

We describe below the configurations, as well as the relevant analytical formulae programmed in the library. The aim of this note is to describe the working principle of the routines, and the basics of magnetostatics are intentionally left to the numerous textbooks and references quoted on the subjects.

Plane, 2-D configurations

Long straight current lines normal to a plane (x,y), or magnetic moments which are uniform in one space dimension and have components only in the (x,y) plane generate magnetic field configurations that are 2-D and plane. The magnetic field only has the two components in the plane (x,y), and the vector potential only has a component normal to the plane, along z. Under this assumption we can make an extensive use of the complex formalism introduced by Beth (Beth, 1966). The position of the current and magnetization sources is defined in the complex plane $\zeta = x + \mathrm{i}\, y$.[1]

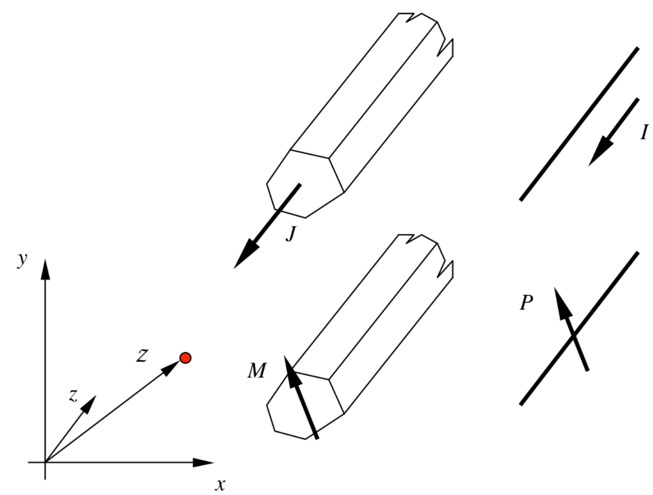

Figure 1. Plane 2-D reference frame and configurations considered for the calculation of magnetic field and vector potential of current and magnetization sources.

As discussed in (Beth, 1966) a complex, analytical function $\mathcal{B} = B_y + \mathrm{i}\, B_x$ can be defined from the x- and y-components of the magnetic field vector. We indicate below the z-component of the current density in a current carrying conductor with J, and the total current with I. By analogy with the definition of the complex field function, we define a complex magnetization function $\mathcal{M} = M_y + \mathrm{i}\, M_x$, where the two components M_y and M_x are the magnetization densities, uniform in z, in a magnetised material. Finally, when computing the field of small magnetic moments, we make use of the complex magnetic moment per unit length $\mathcal{p} = p_y + \mathrm{i}\, p_x$, where the two components p_y and p_x are the magnetic moments per unit length, uniform in z, in the magnetised material.

[1] Note that we indicate with ζ the complex variable, to distinguish it from the axis coordinate z in the general 3-D frame (x,y,z).

The advantage of this definition is that, as $\mathcal{B}$ is analytical, it can be expanded in a series that converges in a circle that does not contain current sources:

$$\mathcal{B} = \sum_{n=1}^{\infty} C_n \left(\frac{\zeta}{R_{ref}} \right)^{n-1} \tag{1}$$

where C_n is called *complex harmonic*, or *multipole coefficient*, of order n, and R_{ref} is a *reference radius* that has a pure normalization function. The complex multipole coefficients have a real and an imaginary part:

$$C_n = B_n + iA_n \quad (2).$$

The real part B_n is the so-called *normal multipole*, while the imaginary part, A_n is the *skew multipole*.

Magnetic field calculation

Filamentary current

The magnetic field generated in a point $\zeta_P = x_P + iy_P$ by a filamentary current I located at $\zeta_C = x_C + iy_C$ is given by:

$$\mathcal{B} = \mu_0 \frac{I}{2\pi\rho} \tag{3}$$

where the distance ρ is defined as:

$$\rho = \zeta_C - \zeta_P \tag{4}$$

Uniform current density with polygonal boundary

We consider the case of a polygonal conductor, delimited by a series of N piecewise straight segments defined in the complex plane through the position of the N vertices ζ_i. A uniform current density J flows inside the polygonal conductor normal to the complex plane, see Fig. 2.

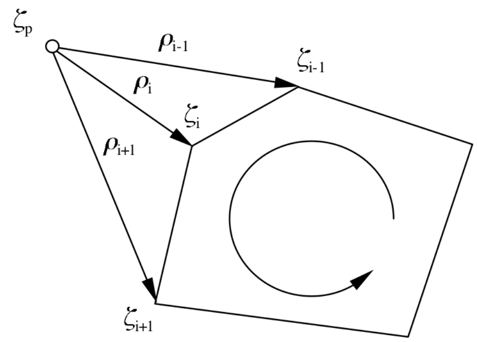

Figure 2. Polygonal conductor with unifom current density in the direction normal to the plane. The field is computed in a point z_p that can be in an arbitrary position in the plane.

The field generated at any point ζ_P in the complex plane is given by the surface integral:

$$\mathcal{B} = \frac{\mu_0 J}{2\pi} \int_S \frac{1}{(\zeta - \zeta_P)} dS \tag{5}$$

The above iontegral can be transformed in the following line integral (Beth, 1966), (Beth, 1967), (Halbach, 1970):

$$\mathcal{B} = \frac{i}{4\pi} \mu_0 J \oint \frac{\zeta^* - \zeta_P^*}{\zeta - \zeta_P} d\zeta \tag{6}$$

where the variable of integration ζ describes the contour of the polygon in positive (anti-clockwise) direction. Introducing the distance vector ρ, as defined in Eq. (4), and recalling that the contour of the polygon between two vertices i-1 and i is a straight line, it is possible to simplify the above integral:

$$\mathcal{B} = \frac{i}{4\pi} \mu_0 J \sum_{i=1}^{N} \int_{\mathbf{r}_{i-1}}^{\mathbf{r}_i} \frac{\rho^*}{\rho} d\rho \tag{7}$$

where we implicitly assume that $\rho_0 = \rho_N$. Introducing the auxiliary quantities:

$$\Delta\rho_i = \rho_i - \rho_{i-1} \tag{8}$$

$$\Delta\rho_i^* = \rho_i^* - \rho_{i-1}^* \tag{9}$$

we can solve the contour integral:

$$\mathcal{B} = \frac{i}{4\pi} \mu_0 J \sum_{i=1}^{N} \left[\Delta\rho_i^* + \frac{\rho_{i-1}^* \Delta\rho_i - \rho_{i-1} \Delta\rho_i^*}{\Delta\rho_i} \left(\ln\rho_i - \ln\rho_{i-1} \right) \right] \tag{10}$$

The expression above provides an exact and closed form solution for the field generated by a polygonal conductor with uniform current density. Its evaluation is straightforward for points ζ_P outside the conductor, but requires care in the monotonic treatment of the argument of the complex natural logarithm for points ζ_P placed inside and on the contour of the conductor[2].

Localised magnetic moment

The magnetic field generated in a point $\zeta_P = x_P + iy_P$ by a localised magnetic moment per unit length $\mathcal{p}$ with components (p_x,p_y) and located at $z_M = x_M + iy_M$ is given by:

$$\mathcal{B} = -\mu_0 \frac{\mathcal{p}^*}{2\pi\rho^2} \tag{11}$$

where the distance ρ is defined similarly to Eq. (4).

$$\rho = \zeta_M - \zeta_P \tag{12}$$

Uniform magnetization with polygonal boundary

As for the magnetic field, we consider the polygonal conductor of Fig. 2. The material in the polygon has a uniform magnetization $\mathcal{M}$ with components (M_x, M_y). The field generated at any point ζ_P in the complex plane is given by the surface integral:

$$\mathcal{B} = -\frac{\mu_0 \mathcal{M}}{2\pi} \int_S \frac{1}{(\zeta - \zeta_P)^2} dS \tag{13}$$

For points external to the polygon, the above integral can be transformed in the following line integral (Beth, 1966; Beth, 1967; Halbach, 1970):

$$\mathcal{B} = -\frac{1}{4\pi i} \mu_0 \mathcal{M} \oint \frac{1}{\zeta - \zeta_P} d\zeta \tag{14}$$

with the same conventions as Eq. (6). For points internal to the polygon the result is the same, but the magnetic field in this case is given by the result of Eq. (14) plus the contribution of the magnetization, μ_0 $\mathcal{M}$. We introduce the distance vector **r**, as defined in Eq. (4), and we write the above integral decomposing over the sides of the polygon:

[2] In general the logarithm of a complex nuber ζ can be written as:
$\log(\zeta) = \log(|\zeta|) + \arg(\zeta)$
while the log of the module of the complex number is well behaved and always single-valued, the argument can flip by 2 π depending on the evaluation method. The care that is needed is to make sure that under all circumstances the argument of the terms in the line integral is a monotonical increasing or decreasing function with no jumps.

$$\mathcal{B} = -\frac{1}{4\pi i}\mu_0\mathcal{M}\sum_{i=1}^{N}\int_{\rho_{i-1}}^{\rho_i}\frac{1}{\rho}\,d\rho \qquad (15).$$

where we implicitly assume that $\rho_0 = \rho_N$. Introducing the auxiliary quantities:

$$\Delta\rho_i = \rho_i - \rho_{i-1} \qquad (16)$$

$$\Delta\rho_i^* = \rho_i^* - \rho_{i-1}^* \qquad (17)$$

we can solve the contour integral:

$$\mathcal{B} = -\frac{1}{4\pi i}\mu_0\mathcal{M}\sum_{i=1}^{N}\left[\frac{\Delta\rho_i^*}{\Delta\rho_i}\left(\ln\rho_i - \ln\rho_{i-1}\right)\right] \qquad (18).$$

The expression above but requires the same care in the monotonic treatment of the argument of the complex natural logarithm for points ζ_P placed inside and on the contour of the conductor as Eq. (10).

Vector potential calculation

Filamentary current

The vector potential generated in a point $\zeta_P = x_P + iy_P$ by a filamentary current I located at $\zeta_C = x_C + iy_C$ is given by:

$$A_z = -\mathrm{Re}\left\{\mu_0\frac{I}{2\pi}\left(\ln\rho + \frac{1}{2}\right)\right\} \qquad (19)$$

where the distance ρ is defined as in Eq. (4).

Uniform current density with polygonal boundary

The vector potential generated by a polygonal conductor as in Fig. 2.2, carrying a uniform current distribution is given by the following surface integral:

$$A_z = -\frac{1}{4\pi}\mu_0 J\int_S\left[\ln\left(\zeta - \zeta_P\right) + \ln\left(\zeta^* - \zeta_P^*\right)\right]dS \qquad (20).$$

that can be transformed (Halbach, 1970) in the line integral:

$$A_z = \frac{i}{8\pi}\mu_0 J\oint\left(\zeta^* - \zeta_P^*\right)\left[\ln\left(\zeta - \zeta_P\right) + \ln\left(\zeta^* - \zeta_P^*\right)\right]d\zeta \qquad (21).$$

As previously, we introduce the distance vector ρ as in Eq. (4), and we break the integral in a summation along the straight sides of the polygon:

$$A_z = \frac{i}{8\pi}\mu_0 J \sum_{i=1}^{N} \int_{\mathbf{r}_{i-1}}^{\mathbf{r}_i} \rho^* \left(\ln\rho + \ln\rho^*\right) d\rho \tag{22}$$

The integral can be solved and leads to the following closed form for the vector potential of a polygonal conductor with uniform current density:

$$\begin{aligned} A_z = \frac{i}{8\pi}\mu_0 J \sum_{i=1}^{N} \Bigg\{ & \left(\frac{\rho_{i-1}^*\Delta\rho_i - \rho_{i-1}\Delta\rho_i^*}{\Delta\rho_i}\right)\left[\rho_i\left(\ln\rho_i - 1\right) - \rho_{i-1}\left(\ln\rho_{i-1} - 1\right)\right] + \\ & \frac{\Delta\rho_i^*}{\Delta\rho_i}\left[\frac{\rho_i^2}{2}\left(\ln\rho_i - \frac{1}{2}\right) - \frac{\rho_{i-1}^2}{2}\left(\ln\rho_{i-1} - -\frac{1}{2}\right)\right] + \\ & \frac{\Delta\rho_i}{\Delta\rho_i^*}\left[\frac{\rho_i^{*2}}{2}\left(\ln\rho_i^* - \frac{1}{2}\right) - \frac{\rho_{i-1}^{*\,2}}{2}\left(\ln\rho_{i-1}^* - -\frac{1}{2}\right)\right]\Bigg\} \end{aligned} \tag{23}$$

that can be evaluated with the same precautions taken for the magnetic field (see previous sections).

Localised magnetic moment

The vector potential generated in a point $\zeta_P = x_P + iy_P$ by a localised magnetic moment $\wp$ with components (p_x, p_y) and located at $\zeta_m = x_m + iy_m$ is given by:

$$A_z = -\mathrm{Re}\left\{\mu_0 \frac{\wp^*}{2\pi\rho}\right\} \tag{24}$$

where the distance ρ is defined as in Eq. (12).

Uniform magnetization with polygonal boundary

The vector potential generated by a polygonal distribution of uniform magnetization, with the polygon defined as in Fig. 2.2, is given by the following surface integral:

$$A_z = -\mathrm{Re}\left\{\mu_0 \frac{\wp^*}{2\pi}\int_S \frac{1}{\left(\zeta - \zeta_P\right)} dS\right\} \tag{25}$$

Apart for a constant, the term in parentheses, and in particular the integral, is the same as that of the magnetic field of a uniform current density in a polygon, Eq. (10). Hence we can use the same procedure, and the final result is:

$$A_z = -\mathrm{Re}\left\{\frac{i}{4\pi}\mu_0 \wp^* \sum_{i=1}^{N}\left[\Delta\rho_i^* + \frac{\rho_{i-1}^*\Delta\rho_i - \rho_{i-1}\Delta\rho_i^*}{\Delta\rho_i}\left(\ln\rho_i - \ln\rho_{i-1}\right)\right]\right\} \tag{26}$$

Harmonics calculation

Filamentary current

A line of current I located at a position $\zeta_C = x_C + iy_C$ in the complex plane generates the following harmonics:

$$C_n = -\mu_0 \frac{I}{2\pi R_{ref}} \left(\frac{R_{ref}}{\zeta_C} \right)^n \qquad (27).$$

Localised magnetic moment

A localised magnetic moment p with components (p_x,p_y) and located at a position $\zeta_m = x_m + iy_m$ in the complex plane generates the following harmonics:

$$C_n = -n\mu_0 \frac{p^*}{2\pi R_{ref}^2} \left(\frac{R_{ref}}{\zeta} \right)^{n+1} \qquad (28).$$

Axisymmetric, 2-D configurations

In the case of axisymmetric circular filamentary or massive coils, the current lines are normal to the plane (R,z) in cylindrical coordinates, and magnetic moments have components only in the (R,z). Both currents and magnetic moments are constant in the third space dimension θ. The resulting magnetic field has only the two components in the plane (R,z), and the vector potential has only one component normal to the (R,z) plane, along θ. We limit ourselves to filamentary currents or magnetic moments, or to polygons in 2-D plane with constant current density and volume magnetization.

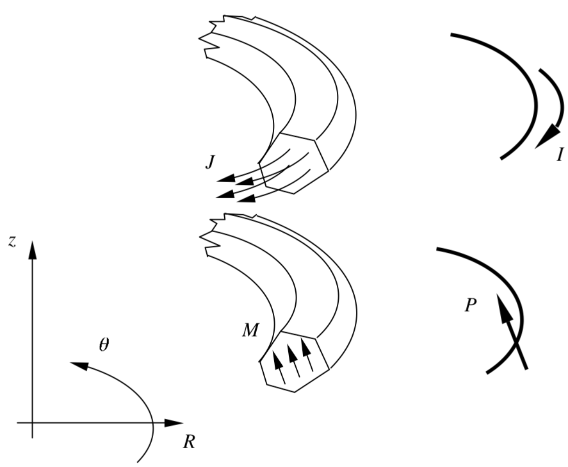

Figure 3. Cylindrical reference frame for the axisymmetric 2-D configuration, and geometries considered for the calculation of magnetic field and vector potential of current and magnetization sources.

The calculation of fields in this configuration invariably results in expression involving the elliptic integrals, and in particular the complete elliptic integral of first kind K and E, defined as follows:

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\left(1 - k^2 \sin^2\theta\right)^{1/2}} \tag{29}$$

$$E(k) = \int_0^{\pi/2} \left(1 - k^2 \sin^2\theta\right)^{1/2} d\theta \tag{30}$$

Magnetic field calculation

Filamentary loop current

The magnetic field generated in a point (R_P,z_P) by a filamentary loop current I with radius R_C and placed at an height z_C is given by:

$$B_R = -\mu_0 \frac{I}{2\pi} \frac{\Delta z}{R_p} \frac{1}{\left[\left(R_C + R_P\right)^2 + \Delta z^2\right]^{1/2}} \left[K(k) - \frac{R_C^2 + R_P^2 + \Delta z^2}{\left(R_C - R_P\right)^2 + \Delta z^2} E(k)\right] \tag{31}$$

$$B_z = \mu_0 \frac{I}{2\pi} \frac{1}{\left[\left(R_C + R_P\right)^2 + \Delta z^2\right]^{1/2}} \left[K(k) + \frac{R_C^2 - R_P^2 - \Delta z^2}{\left(R_C - R_P\right)^2 + \Delta z^2} E(k)\right] \tag{32}$$

where:

$$k^2 = \frac{4 R_C R_P}{\left(R_C + R_P\right)^2 + \Delta z^2} \tag{33}$$

and the distance Δz is defined as:

$$\Delta z = z_P - z_C \tag{34}$$

Uniform current density loop with rectangular boundary

NOTE: This field primitive is presently not documented

Uniform current density loop with polygonal boundary

NOTE: This field primitive is presently not documented

Localised magnetic moment loop

NOTE: This field primitive is presently not documented

Uniform magnetization loop with rectangular boundary

NOTE: This field primitive is presently not documented

Uniform magnetization loop with polygonal boundary

NOTE: This field primitive is presently not documented

Vector potential calculation

Filamentary current loop

The vector potential generated in a point (R_P,z_P) by a filamentary loop current I with radius R_C and placed at an height z_C is given by:

$$A_\phi = \mu_0 \frac{I}{k\pi}\left(\frac{R_c}{R_p}\right)^{1/2}\left[\left(1-\frac{k^2}{2}\right)K(k)-E(k)\right] \tag{35}$$

where k is defined as in Eq. (33).

Uniform current density loop with rectangular boundary

NOTE: This field primitive is presently not documented

Uniform current density loop with polygonal boundary

NOTE: This field primitive is presently not documented

Localised magnetic moment loop

NOTE: This field primitive is presently not documented

Uniform magnetization loop with rectangular boundary

NOTE: This field primitive is presently not documented

Uniform magnetization loop with polygonal boundary

NOTE: This field primitive is presently not documented

3-D configurations

In the case of general, 3-D configurations, the current and magnetization vectors can have arbitrary orientation in the (x, y, z) space. The resulting magnetic field and vector potential also have three non-zero components.

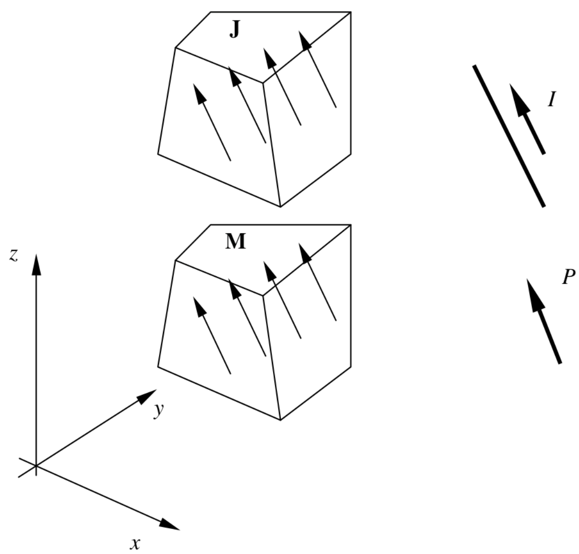

Figure 4. Reference frame for the 3-D configuration, and geometries considered for the calculation of magnetic field and vector potential of current and magnetization sources.

For 3-D configurations we restrict the library of modelling elements to the case of localised sources (filament current I [A] or point-like magnetic moment $\mathbf{P}$ [Am^2]), and to a general volume element with plane faces and constant current density $\mathbf{J}$ [A/m2] or constant volume magnetization $\mathbf{M}$ [A/m]. Furthermore, although the volume element could have an arbitrary number of nodes (the equations below reflect this case), the implementation has been done only for a 8-node hexahedron, which is the most useful modelling element for coil winding packs. To simplify the notation, it is useful to define local reference frames. In the case of a current filament or a localised magnetic moment we define a local reference frame (x',y',z') oriented such that the z' axis has the direction of the current, or of the magnetic moment, and centered in the center of the current filament, or in the location of the magnetic moment. The orientation of the other two axes, x' and y', is inessential, as the results are invariant for a rotation around z'. This reference frame is schematically shown in Fig. 5.

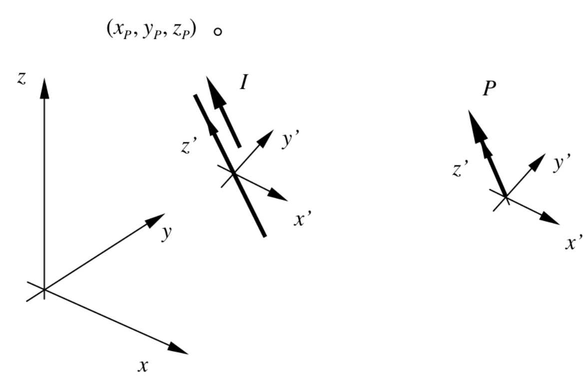

Figure 5. Local reference frames on faces and sides of a source volume.

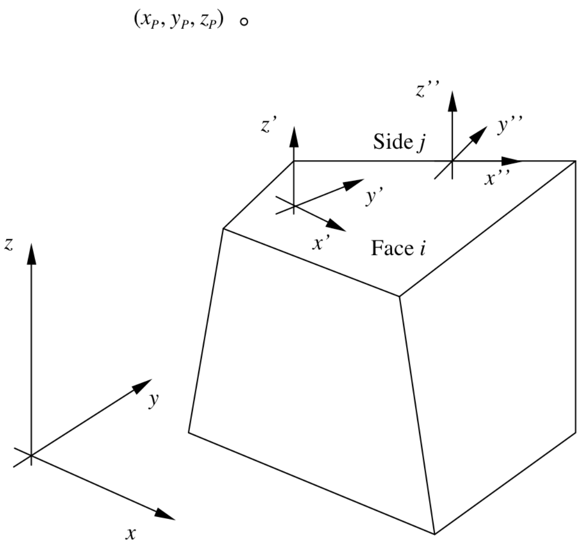

Figure 6. Local reference frames on faces and sides of a source volume.

In the case of a volume element we define two frames, one on the plane faces and one on the sides of the element. For the *i*-th plane face delimiting the volume we define the local reference frame (x',y',z') with z' axis oriented towards the external of the element. This reference frame is shown schematically in Fig. 6. Note that with this choice the z' component of the distance vector between a field calculation point and a source point on the face is a constant.

In turn, each face is delimited by straight sides. For the *j*-th side we define a local reference frame (x'',y'',z'') with z'' axis parallel to z' defined above, and y'' parallel to the side. This reference frame is also shown schematically in Fig. 6. Note that with this choice the y'' component of the distance vector between a field calculation point and a source point on the face is a constant. Furthermore, the equations describing the side becomes $y''=0$ and $z''=0$.

In general, the versors in the direction of the axes of the local reference frames ((x',y',z') or (x'',y'',z'')) are indicated as **t**, **s**, and **n** respectively. The transformation of a vector **g** from the cartesian frame (x, y, z) to the cartesian frame (t, s, n) is obtained by the following matrix relations:

$$\mathbf{g}_{(x,y,z)} = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \begin{bmatrix} t_x & s_x & n_x \\ t_y & s_y & n_y \\ t_z & s_z & n_z \end{bmatrix} \begin{bmatrix} g_t \\ g_s \\ g_n \end{bmatrix} = \mathbf{T}\mathbf{g}_{(t,s,n)} \tag{36}$$

$$\mathbf{g}_{(t,s,n)} = \begin{bmatrix} g_t \\ g_s \\ g_n \end{bmatrix} = \begin{bmatrix} t_x & t_y & t_z \\ s_x & s_y & s_z \\ n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \mathbf{T}^{-1}\mathbf{g}_{(x,y,z)} \tag{37}$$

Magnetic field calculation

Filamentary current

The magnetic field generated in a point (x_P,y_P,z_P) by a straight filament of current I with extremes (x_{C1}, y_{C1}, z_{C1}) and (x_{C2}, y_{C2}, z_{C2}) is given in the local coordinate frame (x', y', z') oriented along the filament and shown in Fig. 5:

$$B_{x'} = -\mu_0 \frac{I}{4\pi} \frac{y'_p}{x'^2_p + y'^2_p} \left[\frac{\frac{L}{2} - z'_p}{\sqrt{x'^2_p + y'^2_p + \left(\frac{L}{2} - z'_p\right)^2}} + \frac{\frac{L}{2} + z'_p}{\sqrt{x'^2_p + y'^2_p + \left(\frac{L}{2} + z'_p\right)^2}} \right] \quad (38)$$

$$B_{y'} = \mu_0 \frac{I}{4\pi} \frac{x'_p}{x'^2_p + y'^2_p} \left[\frac{\frac{L}{2} - z'_p}{\sqrt{x'^2_p + y'^2_p + \left(\frac{L}{2} - z'_p\right)^2}} + \frac{\frac{L}{2} + z'_p}{\sqrt{x'^2_p + y'^2_p + \left(\frac{L}{2} + z'_p\right)^2}} \right] \quad (39)$$

$$B_{z'} = 0 \quad (40)$$

where L is the total length of the current filament, and x_P', y_P', z_P' are the coordinates of the field point in the local reference frame (x', y', z').

The components of the field in the global reference frame (x, y, z) are obtained by rotation of the above values using Eq. (36).

Uniform current density in a solid volume

The magnetic field generated by a current density distribution in an arbitrary volume V is given by:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J} \times \mathbf{r}}{r^3} dV \quad (41)$$

where $\mathbf{r}$ is the vector from the source point (x_Q, y_Q, z_Q) to the field point (x_P, y_P, z_P):

$$\mathbf{r} = \begin{bmatrix} x_P - x_Q & y_P - y_Q & z_P - z_Q \end{bmatrix} \quad (42)$$

We make the hypothesis that the current density is constant in the volume, and that the volume is delimited by plane faces. In this case the calculation of (41) can be performed analytically using the procedure devised in (Collie, 1976) to first reduce volume integrals to surface integrals, and then reduce surface integrals to line integrals. We sketch here the process, without entering in the details of the method described in the above reference.

As the current density is constant, the integral in Eq. (4.6) can be written:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\mathbf{J} \times \int_V \frac{\mathbf{r}}{r^3} dV \tag{43}$$

The vector in the integral can be written as follows[3]:

$$\frac{\mathbf{r}}{r^3} = -\nabla\left(\frac{1}{r}\right) \tag{44}$$

If we indicate with S the surface bounding the volume V, and with $\mathbf{n}$ its normal pointing towards the outside of the element, we can use the following property of vector functions:

$$\int_V \nabla g \; dV = \int_S g\mathbf{n} \; dS \tag{45}$$

which relates the volume integral of the divergence of $\mathbf{g}$ to the flux of $\mathbf{g}$ over the surface S. We can transform the integral in Eq. (4.8) as follows:

$$\mathbf{B} = -\frac{\mu_0}{4\pi}\mathbf{J} \times \left\{ \sum_i \mathbf{n}_i \int_{S_i} \frac{1}{r} \; dS \right\} \tag{46}$$

Above we have indicated with S_i the i-th plane surface composing the boundary of the volume element, and with $\mathbf{n}_i$ its normal (constant on the surface S_i). As the integral is invariant to a change of reference frame, we can choose that each surface integral is performed on the local reference frame (x', y', z') shown in Fig. 6. In this frame a point on the surface S_i has coordinates (x_Q', y_Q', 0) , and the field point (x_P', y_P', z_P') is hence at a constant distance along z'.

We can reduce further the dimension of the integral by using the following vector relation:

$$\int_S \nabla \cdot \mathbf{g} \; dS = \int_l \mathbf{g} \cdot \mathbf{s} \; dl \tag{47}$$

where l is the curve enclosing the surface S, $\mathbf{s}$ is the normal to the curve, in the plane of S, pointing towards the outside of the surface, and the divergence is intended as taken in the plane (i.e. no derivative in the direction $\mathbf{n}$ normal to S). To apply Eq. (47) we use the identity:

$$\frac{1}{r} = \nabla \cdot \left(\frac{\mathbf{r}'}{r + |z'_P|} \right) \tag{48}$$

[3] the gradient is taken with respect to the source point (i.e. the running variable in the integration).

where we have defined the vector **r′** with components:

$$\mathbf{r}' = \begin{bmatrix} x'_P - x'_Q & y'_P - y'_Q & 0 \end{bmatrix} \tag{49}$$

Using Eqs. (47) and (48), we have that Eq. (45) yields:

$$\mathbf{B} = -\frac{\mu_0}{4\pi}\mathbf{J}\times\left\{\sum_i \mathbf{n}_i \left(\sum_j \int_{l_j} \frac{\mathbf{r}'}{r+|z'_P|}\,\mathbf{s}_j\; dl\right)\right\} \tag{50}$$

Above we have indicated with l_j the j-th straight line of the boundary of the surface S_i, and with $\mathbf{s}_i$ its normal (constant on the line l_i). We choose now to integrate in the local reference frame (x'', y'', z'') shown in Fig. 6. In this frame a point on the line l_i has coordinates (x_Q'', 0, 0) , and the field point (x_P'', y_P'', z_P'') is hence at a constant distance both along y'' and z''. Furthermore, the product **r′** $\mathbf{s}_j$ is the distance of the field point along y'', and we can hence write:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\mathbf{J}\times\left\{\sum_i \mathbf{n}_i \left(\sum_j y''_P \int_{l_j} \frac{1}{r+|z'_P|}\; dl\right)\right\} \tag{51}$$

The last step is to solve the line integral above. The general solution is:

$$\int \frac{1}{r+|z|}dx = \ln(x+r) + \frac{|z|}{y}\left(tg^{-1}\left(\frac{x|z|}{yr}\right) - tg^{-1}\left(\frac{x}{y}\right)\right) = I_1(x,y,z) \tag{52}$$

where we have indicated with

$$r^2 = x^2 + y^2 + z^2 \tag{53}$$

We can use the above result to write the following expression for the magnetic field:

$$\mathbf{B} = -\frac{\mu_0}{4\pi}\mathbf{J}\times\left\{\sum_i \mathbf{n}_i \left(\sum_j y''_P \left[I_1\left(x''_{Q2} - x''_P, y''_P, z''_P\right) - I_1\left(x''_{Q1} - x''_P, y''_P, z''_P\right)\right]\right)\right\} \tag{54}$$

where we have indicated with x''_{Q1} and x''_{Q2} the coordinates of the beginning and end of the line l_j, and we have made use of the fact that in the reference frames (x', y', z') and (x'', y'', z'') the z-coordinate of the fild point is the same, i.e. we have $z''_P = z'_P$.

Localised magnetic moment

The magnetic field generated in a point (x_P,y_P,z_P) by a localised magnetic moment **P** located at (x_M, y_M, z_M) is given in the local coordinate frame (x', y', z') oriented along the magnetic moment and shown in Fig. 5:

$$B_\xi = \frac{3\mu_0 P}{4\pi} \frac{x'_P z'_P}{\left(x'^2_P + y'^2_P + z'^2_P\right)^{5/2}} \tag{55}$$

$$B_\eta = \frac{3\mu_0 P}{4\pi} \frac{y'_P z'_P}{\left(x'^2_P + y'^2_P + z'^2_P\right)^{5/2}} \tag{56}$$

$$B_\zeta = \frac{\mu_0 P}{4\pi} \frac{2z'^2_P - x'^2_P - y'^2_P}{\left(x'^2_P + y'^2_P + z'^2_P\right)^{5/2}} \tag{57}$$

where P is the module of the magnetic moment and x_P', y_P', z_P' are the coordinates of the field point in the local reference frame (x', y', z').

The components of the field in the global reference frame (x, y, z) are obtained by rotation of the above values using Eq. (36).

Uniform magnetization in a solid volume

The magnetic field generated by a uniform magnetization in a volume can be obtained in different ways. We choose here the following expression:

$$\mathbf{B} = \begin{cases} \mu_0 \mathbf{M} - \dfrac{\mu_0}{4\pi} \mathbf{M} \cdot \nabla \displaystyle\int_V \frac{\mathbf{r}}{r^3} dV & \text{inside the volume V} \\ -\dfrac{\mu_0}{4\pi} \mathbf{M} \cdot \nabla \displaystyle\int_V \frac{\mathbf{r}}{r^3} dV & \text{outside the volume V} \end{cases} \tag{58}$$

where we used common conventions in vector calculus, and in particular the definition of the operator:

$$\mathbf{M} \cdot \nabla = M_x \frac{\partial}{\partial x} \circ + M_y \frac{\partial}{\partial y} \circ + M_z \frac{\partial}{\partial z} \circ \qquad (59).$$

Examining Eq. (58) we see that the volume integral is the same as already solved for the magnetic field, i.e. Eq. (43). It is hence possible to use the result obtained there, and in particular Eq. (54), to write:

$$-\frac{\mu_0}{4\pi}\mathbf{M}\cdot\nabla\int_V \frac{\mathbf{r}}{r^3}\,dV =$$

$$= -\frac{\mu_0}{4\pi}\mathbf{M}\cdot\nabla\left\{\sum_i \mathbf{n}_i\left(\sum_j y''_P\left[I_1\left(x''_{Q2}-x''_P, y''_P, z''_P\right)-I_1\left(x''_{Q1}-x''_P, y''_P, z''_P\right)\right]\right)\right\} \tag{60}$$

The gradient is intended to be taken with respect to the field point. The derivation of the expression under the summations can be simplified by recalling that:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x''}\frac{\partial x''}{\partial x} + \frac{\partial}{\partial y''}\frac{\partial y''}{\partial x} + \frac{\partial}{\partial z''}\frac{\partial z''}{\partial x} = t_x\frac{\partial}{\partial x''} + s_x\frac{\partial}{\partial y''} + n_x\frac{\partial}{\partial z''} \tag{61}$$

where we used the components of the versors **t**, **s**, **n** in the frame (x, y, z) as defined in Eqs. (36) and (37).

It is hence possible to derive the expression under the summation in the local frame (x'', y'', z'') and transform the derivatives in the frame (x, y, z) using Eq. (61) and analogous for the other directions to solve Eq. (60).

Vector potential calculation

Filamentary current

The vector potential generated in a point (x_P, y_P, z_P) by a straight filament of current I with extremes (x_{C1}, y_{C1}, z_{C1}) and (x_{C2}, y_{C2}, z_{C2}) is given in the local coordinate frame (x', y', z') oriented along the filament and shown in Fig. 5:

$$A_{x'} = 0 \tag{62}$$

$$A_{y'} = 0 \tag{63}$$

$$\begin{aligned} A_{z'} = \mu_0\frac{I}{4\pi}\Bigg[&\ln\left(z'_p+\frac{L}{2}+\sqrt{x'^2_p+y'^2_p+\left(z'_p+\frac{L}{2}\right)^2}\right) \\ &-\ln\left(z'_p-\frac{L}{2}+\sqrt{x'^2_p+y'^2_p+\left(z'_p-\frac{L}{2}\right)^2}\right)\Bigg] \end{aligned} \tag{64}$$

where L is the total length of the current filament, and x_P', y_P', z_P' are the coordinates of the field point in the local reference frame (x', y', z').

The components of the vector potential in the global reference frame (x, y, z) are obtained by rotation of the above values using Eq. (36).

Uniform current density in a solid volume

We use here the same technique as used for the magnetic field calculation. The the vector potential **A** generated by an arbitrary distribution of current **J** in a volume V is:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}}{r} dV \tag{65}$$

where **r** is the vector from the source point (x_Q, y_Q, z_Q) to the field point (x_P, y_P, z_P) defined as in Eq. (42). Under the hypothesis of constant current density, we have:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \mathbf{J} \int_V \frac{1}{r} dV \tag{66}$$

The scalar in the integral can be written as follows:

$$\frac{1}{r} = \frac{1}{2} \nabla \cdot \left(\frac{\mathbf{r}}{r} \right) \tag{67}$$

and in accordance with the relation Eq. (45) we can transform the volume integral of Eq. (67) in the following surface integral:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{J}}{2} \left\{ \sum_i \int_{S_i} \frac{\mathbf{r} \cdot \mathbf{n}_i}{r} \, dS \right\} \tag{68}$$

Above we have indicated with S_i the i-th plane surface composing the boundary of the volume element, and with $\mathbf{n}_i$ its normal (constant on the surface S_i). We choose that each surface integral is performed on the local reference frame (x', y', z') shown in Fig. 6. In this frame the scalar product $\mathbf{r}\,\mathbf{n}_i$ is constant, and is equal to z_P', the z' component of the distance of the field point from the surface.

Based on this, the surface integral becomes:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \mathbf{J} \left\{ \frac{1}{2} \sum_i z'_P \int_{S_i} \frac{1}{r} \, dS \right\} \tag{69}$$

We note now that the surface integral in Eq. (69) is identical to that already solved for the magnetic field generated by a uniform current density in a volume element, in Eq. (46). We can use the procedure already detailed there to obtain the following summation of line integrals in the local reference frame (x'', y'', z''):

$$\mathbf{A} = \frac{\mu_0}{4\pi} \mathbf{J} \left\{ \frac{1}{2} \sum_i z'_P \left(\sum_j y''_P \int_{l_j} \frac{1}{r + |z'_P|} \, dl \right) \right\} \tag{70}$$

Finally, using the definition of the primitive in Eq. (52), we have:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\mathbf{J}\left\{\frac{1}{2}\sum_i z'_P\left(\sum_j y''_P\left[I_1\left(x''_{Q_2}-x''_P, y''_P, z''_P\right)-I_1\left(x''_{Q_1}-x''_P, y''_P, z''_P\right)\right]\right)\right\} \tag{71}$$

where we have indicated with x''_{Q1} and x''_{Q2} the coordinates of the beginning and end of the line l_j, and we have made use of the fact that in the reference frames (x', y', z') and (x'', y'', z'') the z-coordinate of the fild point is the same, i.e. we have $z''_P = z'_P$.

Localised magnetic moment

The vector potential generated in a point (x_P,y_P,z_P) by a localised magnetic moment **P** located at (x_M, y_M, z_M) is given in the local coordinate frame (x', y', z') oriented along the magnetic moment and shown in Fig. 5:

$$A_{x'} = -\frac{\mu_0 P y'_P}{4\pi\left(x'^2_P + y'^2_P + z'^2_P\right)} \tag{72}$$

$$A_{y'} = \frac{\mu_0 P x'_P}{4\pi\left(x'^2_P + y'^2_P + z'^2_P\right)} \tag{73}$$

$$A_{z'} = 0 \tag{74}$$

The vector potential generated in a point (x_P,y_P,z_P) by a localised magnetic moment **P** located at (x_M, y_M, z_M) is given in the local coordinate frame (ξ, η, ζ) oriented along the magnetic moment:
where P is the module of the magnetic moment and x_P', y_P', z_P' are the coordinates of the field point in the local reference frame (x', y', z').

The components of the vector potential in the global reference frame (x, y, z) are obtained by rotation of the above values using Eq. (36).

Uniform magnetization in a solid volume

The the vector potential **A** generated by an arbitrary distribution of magnetization **M** in a volume V is:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{M}\times\mathbf{r}}{r^3}dV \tag{75}$$

where **r** is the vector from the source point (x_Q, y_Q, z_Q) to the field point (x_P, y_P, z_P) defined as in Eq. (42). Under the hypothesis of constant magnetization density, we have:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\mathbf{M} \times \int_V \frac{\mathbf{r}}{r^3} dV \tag{76}$$

The above integral is identical to the expression for the magnetic field generated by a constant current density in a volume (see Eq. (43)), where the vector potential takes the place of the magnetic field and the magnetization takes the place of the current density. Under the hypothesis of a volume delimited by plane faces, and using the local coordinate frame (x'', y'', z'') as shown in Fig. 6, we have:

$$\mathbf{A} = -\frac{\mu_0}{4\pi}\mathbf{M} \times \left\{ \sum_i \mathbf{n}_i \left(\sum_j y''_P \left[I_1\left(x''_{Q2} - x''_P, y''_P, z''_P\right) - I_1\left(x''_{Q1} - x''_P, y''_P, z''_P\right)\right]\right)\right\} \tag{77}$$

where the primitive I_1 is defined in Eq. (52), and we have indicated with x''_{Q1} and x''_{Q2} the coordinates of the beginning and end of the line l_j.

References

(Beth, 1966) R.A. Beth, *Complex Representation and Computation of Two-Dimensional Magnetic Fields*, J. Appl. Phys, **37** (7), 2568-2571, 1966.

(Beth, 1967) R.A. Beth, *An Integral Formula for Two-Dimensional Fields*, J. Appl. Phys, **38** (12), 4689-4692, 1967.

(Collie, 1976) C.J. Collie, Magnetic Fields and Potentials of Linearly Varying Current or Magnetisation in a Plane Bounded Region, Proceedings of Compumag Conference, Oxford, 86-95, 1976

(Halbach, 1970) K. Halbach, *Fields and First Order Perturbation Effects in Two-Dimensional Conductor Dominated Magnets*, Nucl. Inst. and Meth., **78**, 185-198, 1970.