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Analytical Calculation of Vector Potential in an Isoparametric Brick

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Distribution: Internal

Summary

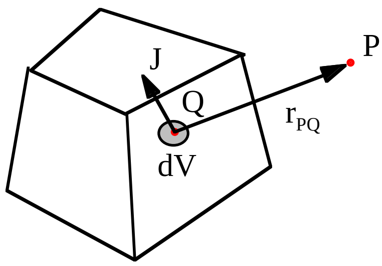
We describe the calculation algorithm for vector potential generated by an isoparametric brick with uniform current density. The calculation is fully analytic and stable inside and outside the brick.

1. Introduction

The calculation of vector potential generated by a current distribution is of interest, especially for the evaluation of self and mutual inductances. In particular we wish to show here how it is possible to compute the vector potential generated by an 8-nodes isoparametric brick with flat faces. We start with the definition of the vector potential A generated by an arbitrary distribution of current:

$$A = \frac{\mu_0}{4\pi} \iiint_V \frac{J}{r_{PQ}} dV \qquad (1.1)$$

where J is the current density in the volume V and r_{PQ} is the module of the distance between a point Q centered in the volume element dV, and the point P where we wish to compute the vector potential. We show this situation below.



It is useful to recall two fundamental properties of vector functions **g**:

$$\iiint_V \nabla \mathbf{g} dV = \iint_S \mathbf{gn} dS \tag{1.2}$$

$$\iint_S \nabla_s \mathbf{g} dS = \oint_\Gamma \mathbf{gt} d\Gamma \tag{1.3}$$

namely the volume integral of the divergence of **g** equals the flux of **g** over the surface S (with normal **n**) bounding the volume, and the surface integral of the *surface* divergence of **g** equals the line integral of **g** along a line Γ bounding the surface, with tangent versor **t**. In the following section we show how using Eqs. (1.2) and (1.3) the calculation of vector potential of Eq. (1.1) can be done completely analytically for a brick element with plane faces, a very attractive feature to boost speed and accuracy in numerical calculations.

2. Vector potential calculation

In general we can write that:

$$\frac{1}{r_{pq}} = \frac{1}{2} \nabla \left(\frac{\mathbf{r}_{PQ}}{r_{PQ}} \right) \tag{2.1}$$

and in accordance with the two relation above, and Eq. (1.2), the volume integral of Eq. (1.1) can be transformed in a surface integral:

$$A = \frac{\mu_0}{4\pi} \frac{J}{2} \iint_S \frac{\mathbf{r}_{PQ}}{r_{PQ}} \mathbf{n} dS \tag{2.2}$$

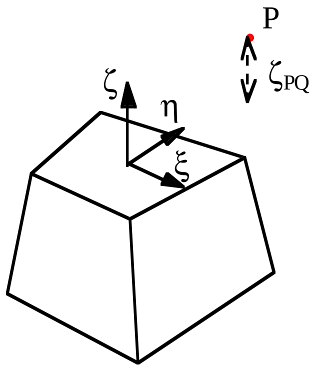
To compute the surface integral of Eq. (2.2) we take an isoparametric brick and we decompose the surface integral in the sum of the integrals of all faces. In addition we choose a reference frame in each face such that the ζ axis has the orientation of the normal **n** (pointing outwards). Because of this choice we will have that (by definition)

$$\mathbf{r}_{PQ} \mathbf{n} = \zeta_{PQ} \tag{2.3}$$

where ζ_{PQ} is the ζ-component of the distance $\mathbf{r}_{PQ}$ and is a constant for the face i. Based on this, the surface integral becomes now:

$$A = \frac{\mu_0 J}{8\pi} \sum_i \left(\zeta_{PQ} \iint_{S_i} \frac{1}{r_{PQ}} \mathbf{n} dS_i \right) \tag{2.4}$$

where the sum extends, for the isoparametric brick to 6 faces.



We now define a function:

$$\mathbf{T}_i = \frac{1}{2} \frac{\xi_{PQ} + \eta_{PQ}}{r_{PQ} + |\zeta_{PQ}|} \tag{2.5}$$

that is such that:

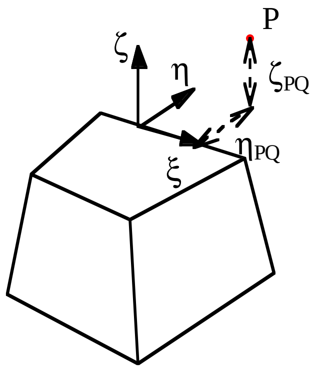
$$\nabla_s \mathbf{T}_i = \frac{\partial \mathbf{T}_i}{\partial \xi} + \frac{\partial \mathbf{T}_i}{\partial \eta} = \frac{1}{r_{PQ}} \quad (2.6).$$

Using then Eq. (2.6) and Eq. (1.3) we can write that each surface integral in the sum of Eq. (2.4) can be decomposed in line integrals as follows:

$$A = \frac{\mu_0 J}{8\pi} \sum_i \left[\zeta_{PQ} \sum_j \left(\int_{\Gamma_{ij}} \mathbf{T}_i \mathbf{t}_j d\Gamma_{ij} \right) \right] \tag{2.7}$$

where index j runs over the single lines Γ_{ij} delimiting each surface S_i. We finally choose the reference frame for the line integrals such that the ξ direction is coincident with the versor **t**, and we get by this choice that

$$\mathbf{T}_i \mathbf{t}_j = \frac{\eta_{PQ}}{r_{PQ} + |\zeta_{PQ}|} \quad (2.8).$$



This furtherw simplification leads to the final form of the integral to be computed:

$$A = \frac{\mu_0 J}{8\pi} \sum_i \left[\zeta_{PQ} \sum_j \left(\eta_{PQ} \int_{\Gamma_{ij}} \frac{1}{r_{PQ} + |\zeta_{PQ}|} d\Gamma_{ij} \right) \right] \qquad (2.9).$$

The line integral is of the general form:

$$\int \frac{1}{\sqrt{x^2 + y^2 + z^2} + |z|} dx = -\frac{|z|}{y} tg^{-1}\left(\frac{x}{y}\right) + \frac{|z|}{y} tg^{-1}\left(\frac{x|z|}{yr}\right) + \ln(x + r) \qquad (2.10)$$

where we defined:

$$r^2 = x^2 + y^2 + z^2 \qquad (2.11).$$

Using the result above, the analytic expression of the vector potential is finally:

$$A = \frac{\mu_0 J}{8\pi} \sum_i \zeta_i \sum_j \eta_j \left\{ \frac{|\zeta_i|}{\eta_j} \left[tg^{-1}\left(\frac{\xi_2 |\zeta_i|}{\eta_j r_2}\right) - tg^{-1}\left(\frac{\xi_1 |\zeta_i|}{\eta_j r_1}\right) + tg^{-1}\left(\frac{\xi_1}{\eta_j}\right) - tg^{-1}\left(\frac{\xi_2}{\eta_j}\right) \right] + \right. \qquad (2.12)$$
$$\left. + \left[\ln(\xi_2 + r_2) - \ln(\xi_1 + r_1) \right] \right\}$$

where we have indicated with subscripts *i* and *j* the distances in ζ and η direction that are constants in the integral, and indices 1 and 2 stand for values of the distance in ξ direction and total distance *r* calculated at the initial and final points in each line (extremes of integration).